Ch11.1 & Ch11.2 (Ch12.1e) Bessel Functions of the 1st Kind & Orthogonality

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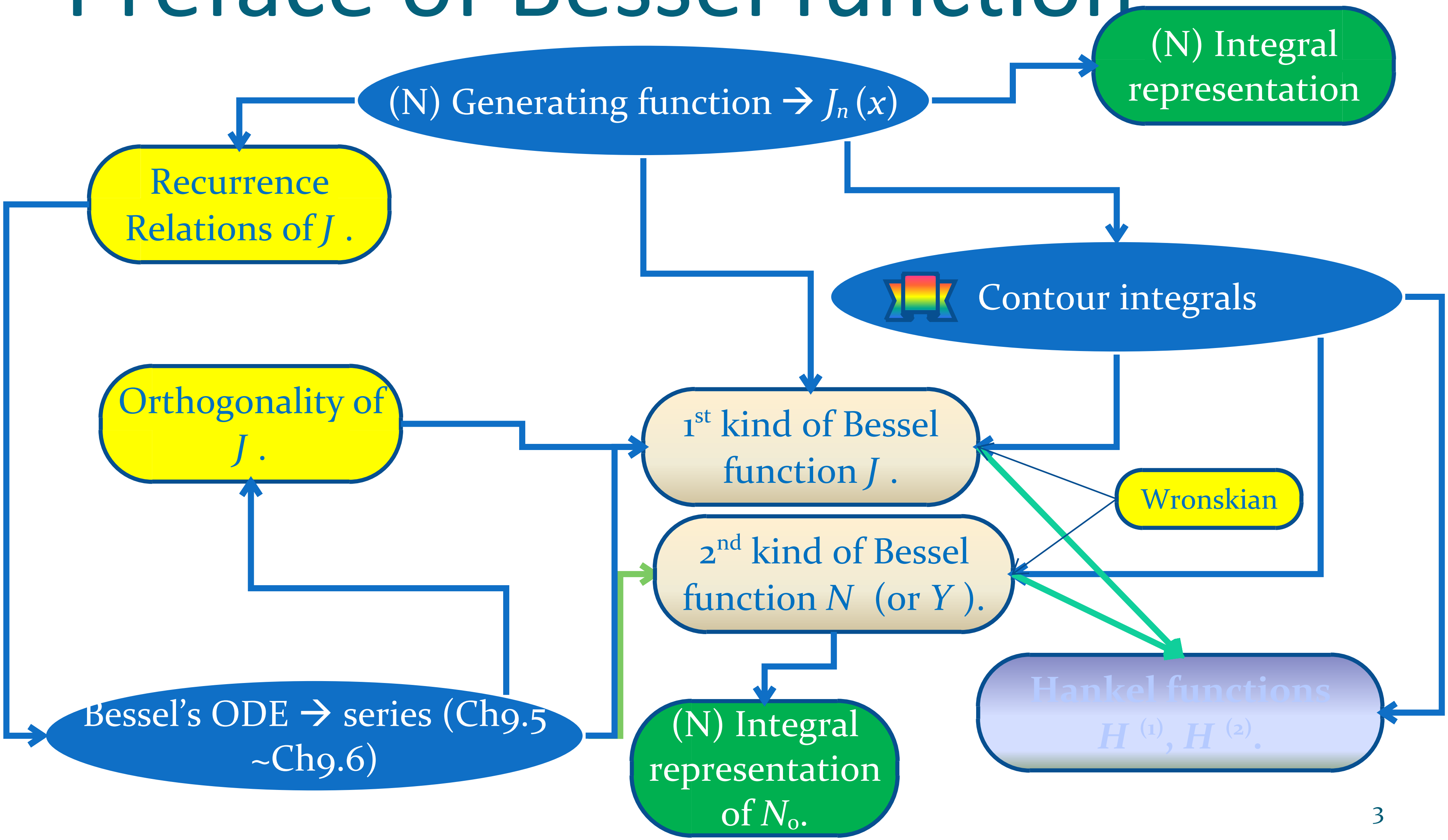
Contents

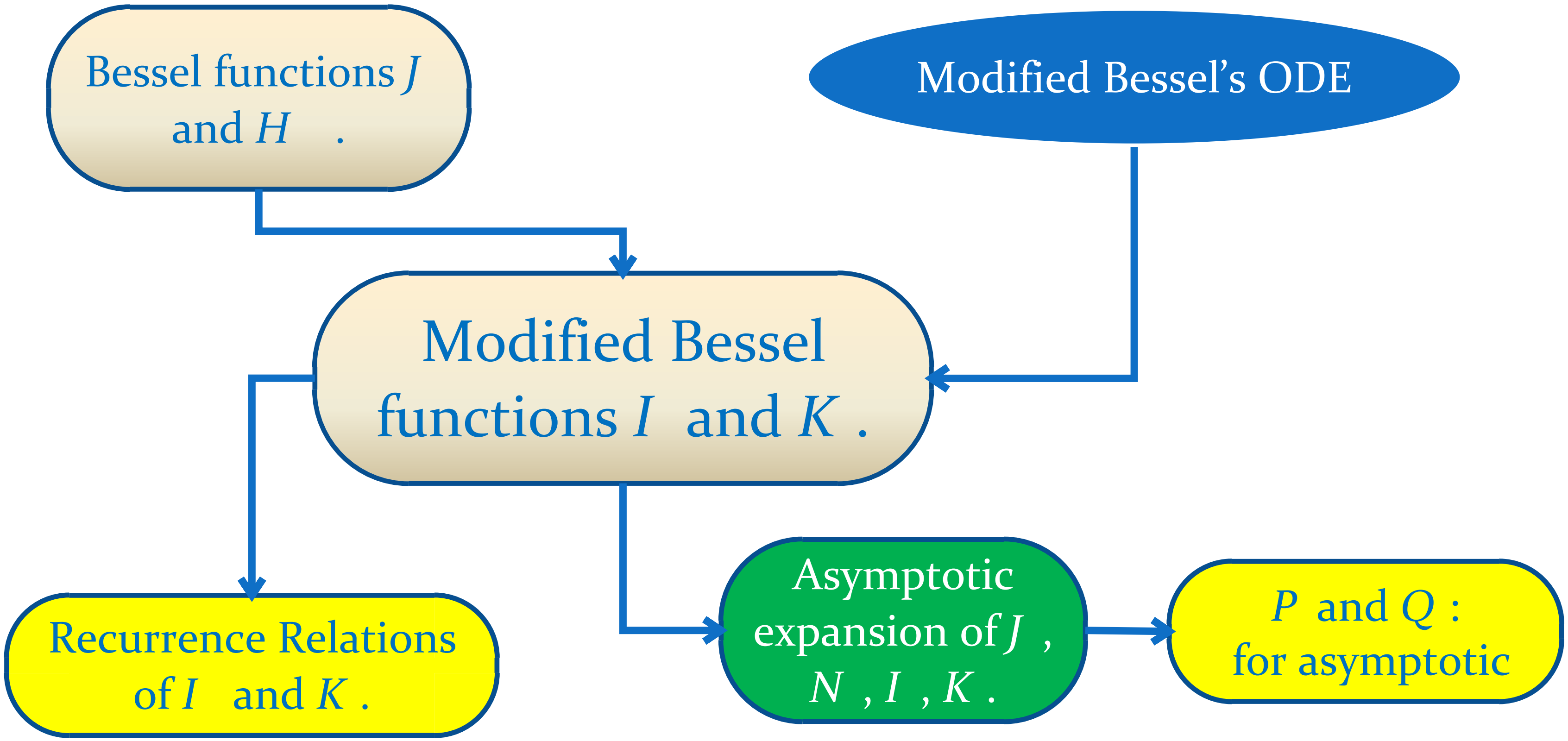
- Reference: http://en.wikipedia.org/wiki/Bessel_function

(N) Means only for n Z.
R.

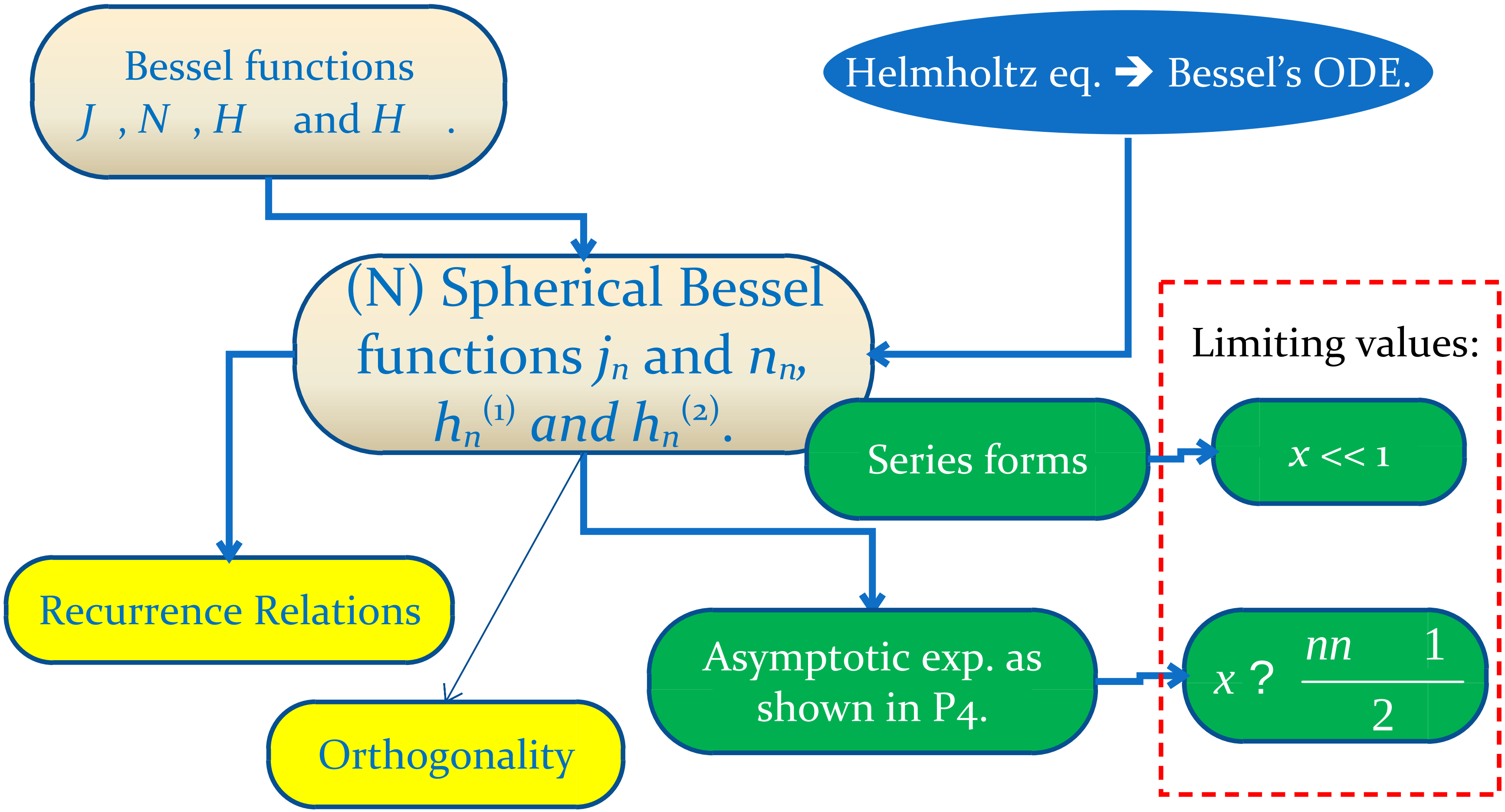
Preface of Bessel function



Preface of Modified Bessel functions



Preface of Spherical Bessel functions



Generating function for integral order (□□□P675~P678)

- (N) Generating function: $g x t e, J x t \quad \frac{x}{2}\; t \;\frac{1}{t} \quad n \quad n \quad n$
  - From this generating function, we can get:
    - $J_n x \quad \frac{1}{nss \;\; !!2} \; \frac{x}{}^{ns2} \; ,1 \quad s\;0$
    - $J_{\;n}(x)=(-1)^n J_n(x)=J_n(-x)$. ------(2)
      - Gotten from $g(x,t)=g(-x,t^{-1})$
    - Recurrence relations:
      - From $_x g=(t-1/t)/2*g= \;\; J'_n(x)t^n$ ➔ $2J'_n=J_{n-1} \;\; J_{n+1}$ --------(3)
        - $J'_0= \;\; J_1$.
      - From $_t g=x/2*g*(1+1/t^2)= \;\; J_n(x)nt^{n\;\;1}$ ➔ $J J J_{nn11} \quad \frac{2n}{x}$. ---(4)
      - From g(0,t)=g(x,1)=1 ➔ $J_0(0)=1, J_n(0)=0$ & $1=J_0(x)+2 \;\; J_{2n}$(x).
    - By Eqs. (3) & (4) we can get Bessel's equation.

Generating function for integral order (□□□P679~P680)

- From $g(u+v,t)=g(u,t)g(v,t)$ ➔ $J_n(u+v)=\sum_l J_l(u)J_{n-l}(v)$. --(5)
- (N) Integral representation:
  - From $g(x,e^{i\theta})=\exp(ix\sin\theta)=\sum J_n(x)\exp(in\theta)$.
  - Therefore,

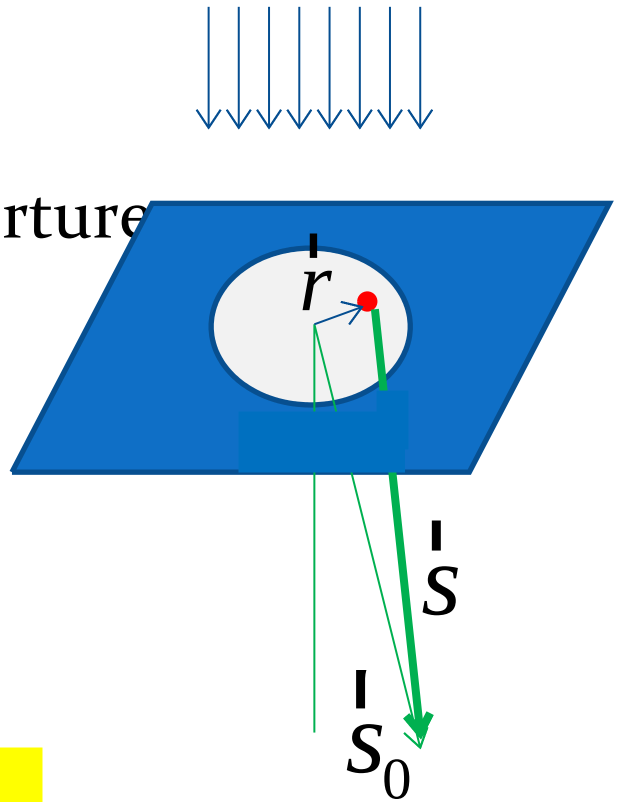
$$J_n(x)=\frac{1}{2\pi}\int_0^{2\pi} e^{ix\sin\theta}e^{-in\theta}\,d\theta=\frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta-n\theta)\,d\theta$$

$$e^{ikr\cos\theta}=\exp\ldots=g(kr,ie^{i\theta})=\sum_n i^n J_n(kr)e^{in\theta}.$$

$$J_0(x)=\frac{1}{2\pi}\int_0^{2\pi} e^{ix\sin\theta}\,d\theta=\frac{1}{2\pi}\int_0^{2\pi} e^{ix\cos\theta}\,d\theta \quad .7$$

Example (□□□P680~P682)

- Fraunhofer Diffraction, Circular Aperture
  - The net light wave will be



$,tC\,\exp\, erdrdiksss^{iksit}\quad 0$

$Cerdrdiksrs^{iksit}\quad \exp\quad |rr_{00}|$

$\sim \exp\, Cerdrdiksr^{iksit}\quad \hat{}_{0}\, g\, r$

$\sim 2\exp\, Csin\, cos\, dikr^{iksit}$

- Based on Eq. (7), we get

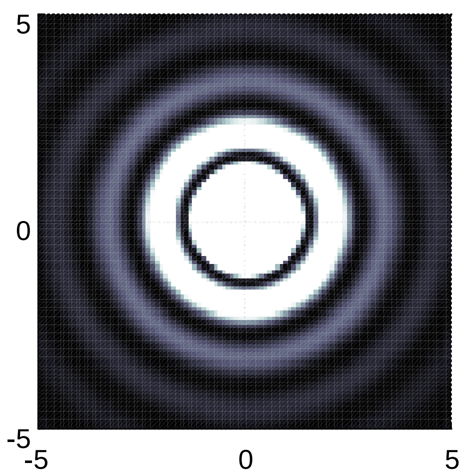
$,tC2\sin\, drJkr^{iksit\, 0}\quad 0\quad 0$

where $\sin\, rdrJrxJxJka_{0}^{a}\ \frac{1}{2}_{011}\quad |_{0}^{a}\quad \frac{a}{k\sin}$

Example (continue)

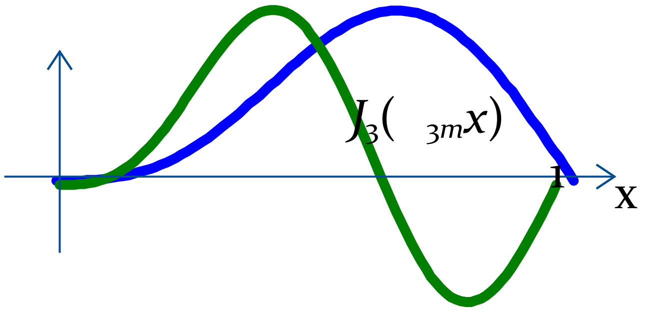
- Therefore, when *ka*sin ~3.8317… , | |→0.
- We get:

$$\sin\sim\sim.\ \frac{3.8317}{2\quad aD}\,1.22\,\frac{}{}$$



Orthogonality (□□□P694~P695)

- $_m$: the m^{th} zero of J .
- $\int_0^1 JxJxxdxC_{mnmn}$?



- Derivation:
  - Bessel Eq. :

$$\frac{dddd}{dxdxxdxdxx} xJxxJxxxJx^{22} \quad 0.0. \quad {}^{2}_{mm}$$

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$\mathbf{L}_m$

$$\int_0^1 JxJxJxJxdx_{nmmmnn} \mathbf{LL}$$

  -

$$^{22}_{mnmnnm} \quad CxJxxJx \quad \frac{dJxdJx_{mn}}{dxdx} \quad \Big|_{x\ 0}^{x\ 1} \quad 0$$

$$C_{mn} \quad \frac{JJJJ_{mnmnnm} \quad ''}{^{22}_{nm}} \quad \text{if } 1.$$

Why?

Orthogonality (continue)

- When m n, we get C_{mn}=0.
- When m=n, use L'Hospital role:

$$C_{nn} \quad \frac{\dfrac{J_m J_n J_m J_n \quad ''}{\quad m \quad}}{22 \quad \dfrac{nm}{m}} \Bigg|_{mn}$$

$$\frac{{}_{mn}J \quad JJ' \quad {}_{nn} \quad JJ'J \quad {}_{mnnn}}{22 \quad {}_{n}} \qquad \frac{J'^{2}_{n}}{}$$

$$\frac{J_n \quad J_1^{\;2}}{22} \qquad \frac{J^2{}_{1\;n}}{}.$$

Orthogonality (continue)

- Therefore, we get

$$\int_0^1 J x J x x dx_{nmmnmn} \quad \frac{JJ^{22}\;{}_{nn}}{22} \quad \frac{1}{\quad} \quad .$$

- In x [0,1] and $f(x=1)=0$, ${}_m(x)=J$ (${}_m x$) will form a complete set, because **H** is a hermitian operator when these boundary condition are held.

$$\underbrace{\frac{d}{dx}x\frac{d}{dx}x}_{\mathbf{H}} \quad \frac{2}{\quad} \quad {}_{mmm} \quad {}^{2}$$

$$or\, xxxx \underbrace{\frac{11}{\sqrt{xx}}\frac{dd}{dxdx}}_{\mathbf{H}} \quad \frac{2}{\quad} \quad \frac{}{\sqrt{\;}} \quad \sqrt{\;}\, {}_{mmm} \quad {}^{2} \quad \sqrt{\;}$$

Q: However, ${}_m(0)=0$ when 0. Do we need to force the function space obey $f(0)=0$?

Bessel Series (continue)

- A function may be expanded in

$$ffaa \quad \sum_{m\ 1} \frac{JxJxfax_{mm} \quad ,}{JxJx \quad _{mm} \quad ,} \text{,where 0,0,and 1.}$$

$$\sum_{m\ 1} cJ_{mm} \quad \frac{}{a} \; .$$

Homework

- 11.1.1 (12.1.1e)
- 11.1.22 (12.1.14e)
- 11.2.3

Nouns

- **Addition theorem of 1st kind of Bessel function**:

- **Bessel (Fourier-Bessel) Series**: